Quasi-Implicative Lattices and the Logic of Quantum Mechanics

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Mittelstaedt has defined the class of quasi-implicative lattices and shown that an ortholattice (orthocomplemented lattice) is quasi-implicative exactly if it is orthomodular (quasi-modular). He has also shown that the quasi-implication operation is uniquely determined by the quasi-implicative conditions. One of Mittelstaedt's conditions, however, seems to lack immediate intuitive motivation. Consequently, this paper seeks to provide a number of reformulations of the quasi-implicative conditions which are more intuitively plausible. Three sets of conditions are examined, and it is shown that each set of conditions is both necessary and sufficient to ensure that an ortholattice is orthomodular, and each set of conditions uniquely specifies the implication operation to be Mittelstaedt's quasi-implication. Various properties of the quasi-implication are then investigated. In particular, it is shown that the quasi-implication fails to satisfy a number of laws associated with the classical material conditional. Various weakenings of these laws, satisfied by the quasi-implication, are also discussed.

Introduction

Since Birkhoff and von Neumann[1] first suggested that the logic appropriate to the elementary propositions of quantum mechanics (QM) is non-classical, a considerable amount of research has been pursued on the quantum logical approach to the foundations of QM and on the mathematical structures associated with quantum logic (QL). Birkhoff and von Neumann singled out the distributive law of classical logic as suspect, replacing it by the weaker modular law. More recently, even the modular law has been abandoned in favor of the still weaker law of orthomodularity (quasi-modularity[2] or weak modularity[3]). This latter move has resulted from the fact that the lattice of subspaces of Hilbert space, which provides the concrete model for QL, is not modular in the infinite dimensional case although it is orthomodular. To a large extent the study of QL has indeed been subsumed under the study of general orthomodular lattices, and in some cases orthomodular partially ordered sets. This paper, however, concentrates on orthomodular lattices.

From the outset it has seemed natural to regard the quantum lattice as representing a non-classical logic. Nevertheless, serious doubts have been raised concerning whether QL (more generally, orthomodular lattices) can be properly regarded as a logic rather than merely an algebraic structure only analogous to logic properly so called. Doubts along these lines have in particular been raised by Jauch and Piron[4], who argue that interpreting a lattice, or class of lattices, as a logic requires the lattice to admit the definition of the algebraic counterpart of the modus ponens inference scheme. In other words, the lattice must admit an *implication* or *conditional operation* in terms of which such an inference scheme can be incorporated. This adequacy requirement on a logical interpretation of a lattice has also been proposed by Kunsemüller[5] and Mittelstaedt[6, 7]. Of particular interest in this paper is the work of Mittelstaedt who has investigated a number of properties of what he calls *quasi-implicative lattices* and the *quasi-implication operation*[7]. In particular he has shown that an ortholattice (orthocomplemented lattice) is quasi-implicative if and only if it is orthomodular, and that the quasi-implication operation is uniquely determined by the quasi-implicative conditions.

The present paper, which is largely a continuation of Mittelstaedt's work, seeks to provide a number of reformulations of the quasi-implicative conditions, reformulations which are somewhat more intuitively plausible than Mittelstaedt's original conditions. For each set of conditions, it is shown that the set of conditions is both necessary and sufficient to ensure that an ortholattice is orthomodular, and that the set of conditions uniquely specifies the implication operation to be Mittelstaedt's quasi-implication. Besides the quasi-implicative conditions, another set of conditions, called the F-implicative conditions, is investigated. It is shown in all cases in which the lattice in question is orthomodular that the F-implicative conditions uniquely specifies the implication operation to be the quasi-implication. On the other hand, it is shown that not all F-implicative lattices (ortholattices satisfying the F-implicative conditions) are orthomodular. Various properties of the quasi-implication are also investigated; it is shown that the quasi-implication fails to satisfy a number of laws associated with the classical material implication. Various weaker laws are also considered, however, and shown to be satisfied by the quasi-implication.

1. Ortholattices and Orthomodular Lattices

In this section we briefly review the theory of orthomodular lattices. A lattice can be order-theoretically characterized as a partially ordered set $\langle L, \leqq \rangle$ in which every pair of elements, a, b, has both a meet (greatest lower bound or infimum) $a \wedge b$ and a join (least upper bound or supremum) $a \vee b$. A lattice can be equivalently characterized as a set L together with two binary operations, meet $\wedge$, and join $\vee$, satisfying the following axioms.

(L1) $a \wedge b = b \wedge a$

(L2) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(L3) $a \wedge (a \vee b) = a$

(L4) $a \vee b = b \vee a$

(L5) $a \vee (b \vee c) = (a \vee b) \vee c$

(L6) $a \vee (a \wedge b) = a$

The associated partial ordering is then defined so that: $a \leqq b$ iff $a \wedge b = a$; alternatively: $a \leqq b$ iff $a \vee b = b$. A *bounded lattice* is a lattice with two distinguished elements 0 and 1 such that for all $b \in L$, $0 \leqq b \leqq 1$. In a bounded lattice, a *complement* of an element b is any element c such that $b \wedge c = 0$ and $b \vee c = 1$. A *complemented lattice* is a bounded lattice in which every element has at least one complement. An *orthocomplemented lattice*, or *ortholattice*, is a complemented lattice L together with a unary operation $^\perp$: $L \to L$ ($a \mapsto a^\perp$), satisfying the following conditions.

(O1) $a^\perp$ is a complement of a,

(O2) $(a^\perp)^\perp = a$,

(O3) $a \leqq b$ implies $b^\perp \leqq a^\perp$.

As an immediate consequence of (O1)–(O3), an ortholattice satisfies de Morgan's laws.

(dM1) $(a \wedge b)^\perp = a^\perp \vee b^\perp$,

(dM2) $(a \vee b)^\perp = a^\perp \wedge b^\perp$.

Where L is a lattice and $a, b, c \in L$, the triple $\{a, b, c\}$ is *distributive* if and only if every substitution of a, b, c into the following equations is satisfied by L.

(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A *distributive lattice* is then a lattice in which every triple is distributive. A distributive ortholattice is customarily referred to as a *Boolean lattice* or *Boolean algebra*.

There are two weakenings of the distributive laws which are of interest in QL. A *modular lattice* is a lattice in which every triple $\{a, b, c\}$ such that $a \leqq c$ is distributive. This condition is equivalent to the following.

(M) $a \leqq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$.

A further weakening of the distributive laws, also a weakening of the modular law, is the law of *orthomodularity* (weak modularity or quasi-modularity), which can be defined in a number of ways. First of all, on any ortholattice one can define a relation K of compatibility as follows.

(K) aKb iff $a = (a \wedge b) \vee (a \wedge b^\perp)$.

The question immediately arises whether the relation as defined by (K) is symmetric, that is, whether the following obtains.

(S) aKb iff bKa.

This is not the case for all ortholattices, but it can be shown that the relation K on an ortholattice L is symmetric just in case L satisfies the law of orthomodularity[8], which is customarily stated as follows[9].

(OM) $a \leqq b$ implies $b = a \vee (b \wedge a^\perp)$.

Besides condition (OM), the law of orthomodularity can be equivalently characterized by any of the following conditions.

(OM2) $a \leqq b$ implies $a = b \wedge (a \vee b^\perp)$,

(OM3) $a \wedge (a^\perp \vee (a \wedge b)) \leqq b$,

(OM4) $a \leqq b$ and $c \leqq b^\perp$ implies $a = b \wedge (a \vee c)$,

(OM5) $a \leqq b^\perp$ and $a \vee b = 1$ implies $a = b^\perp$,

(OM6) aKb and aKc implies $\{a, b, c\}$ is distributive.

That orthomodular lattices satisfy condition (OM6) was proved independently by Foulis[10] and Holland[9], and is therefore referred to as the Foulis-Holland theorem. This theorem, which will be denoted F-H, is particularly useful in the practical calculations of orthomodular lattice theory. Also useful in practical calculations are the following: aKa; aKb implies $aKb^{\perp}$; $a \leqq b$ implies aKb; aKb and aKc implies $aKb \wedge c$ and $aKb \vee c$. The Foulis-Holland theorem, along with these facts, will be used almost without comment throughout this paper. We also note that as an immediate consequence of the Foulis-Holland theorem, if the relation K is universal on an ortholattice L (i.e., $K = L \times L$), then L is Boolean. Thus what distinguishes general orthomodular lattices from Boolean lattices is the existence in the former of pairs of elements which are not compatible.

Before ending this section it is useful to remark that the lattice of subspaces of Hilbert space, which provides the concrete model for QL, is orthomodular in every case and modular in the finite case. The lattice operations are given as follows. Where S and T are two subspaces, the infimum of S and T is the set-theoretic intersection $S \cap T$; the supremum of S and T is the closed linear span $S \oplus T$, and the orthocomplement of S is the orthogonal complement $S^{\perp}$. Besides being orthomodular, the lattice of subspaces is atomic, complete, and semi-modular[11,3]. In this paper, however, these properties will not be considered essential to QL, which will be regarded as characterized by the class of orthomodular lattices.

2. The Conditional in Quantum Logic

We now consider the problem of introducing into QL an operation which has the essential properties of a conditional or implication connective. Before examining this question it is important to clear up a possible confusion concerning the distinction between the implication *relation* and the conditional or implication *operation*. It is generally agreed that the partial ordering of the lattice of QM can be logically interpreted as a *relation* of implication, more specifically semantic entailment. The relation of semantic entailment is customarily denoted by a double turnstile $\Vdash$, and to say that a proposition P semantically entails Q is to say that whenever P is true so is Q; that is to say, any model (world, state, situation) which makes P true also makes Q true. The implication relation must not, however, be confused with the implication operation which is a logical connective often denoted by the horseshoe $\supset$. Lattice theoretically, the implication operation must be represented by a binary lattice operation (to be denoted $\rightarrow$), on a par with meet and join, such that for every pair of lattice elements a, b the implication operation yields another lattice element $a \rightarrow b$. The logical distinction is obvious. "$a \leqq b$" is a lattice theoretic *assertion* about the elements a, b and represents a statement which logically would appear in the metalanguage. On the other hand, $a \rightarrow b$ is a lattice *element* and represents a statement which logically would appear in the object language. The advantage of having an implication operation as well as an implication relation becomes clear as soon as we attempt to represent lattice theoretically certain classical laws of material implication in which the implication connective is iterated. Peirce's law is an example of such a law.

(P) $\quad ((a \rightarrow b) \rightarrow a) \rightarrow a\,.$

Using the lattice ordering to express this we obtain

(P ?) $\quad ((a \leqq b) \leqq a) \leqq a\,,$

which is not well formed since a binary relation cannot be meaningfully iterated in this way. On the other hand, if the arrow in (P) is interpreted as a binary lattice operation, then the expression is well formed since a binary operation can be meaningfully iterated. If we furthermore wish to assert that Peirce's law is valid for a particular lattice, we write

(P*) $\quad ((a \rightarrow b) \rightarrow a) \rightarrow a = 1\,,$

where 1 is the unit element of the lattice which represents the identically (always) true proposition. Whether (P*) obtains in a given lattice will of course depend on the particular lattice operation assigned to $\rightarrow$.

Obviously not any binary lattice operation will qualify as an implication operation. We must therefore decide what criteria must be satisfied by a binary operation in order to be regarded as an implication operation. First of all, it seems minimally required of any "If ... then" operation that it satisfy the law of modus ponens, which lattice theoretically is given as follows.

(MP) $\quad a \wedge (a \rightarrow b) \leqq b\,.$

Next we should expect that an implication *operation* should be related to the *relation* of implication (semantic entailment) in such a way that if a semantically entails b, then $a \to b$ is valid (true in all situations). Translated into lattice theoretic terms, this requirement becomes the following.

(E) $\quad a \leqq b$ implies $a \to b = 1$.

Note that the converse of (E) is an immediate consequence of (MP). (E) can therefore be strengthened to

(E*) $\quad a \leqq b$ iff $a \to b = 1$.

Conditions (MP) and (E) exhaust the obvious *minimal* criteria for implicationhood, and may be called the *minimal implicative conditions*. They are satisfied by all generally accepted implication connectives, including the classical, intuitionistic, strict, and counterfactual conditionals. At the same time, the extreme generality of conditions (MP) and (E) prevent them from specifying a unique implication operation in any given situation. These conditions must therefore be appropriately strengthened if we wish to characterize a particular operation. For example, in the case of classical and intuitionistic logic, this is accomplished by replacing (E) by

(I) $\quad a \wedge x \leqq b$ implies $x \leqq a \to b$,

which together with condition (MP) is equivalent to

(I*) $\quad a \wedge x \leqq b$ iff $x \leqq a \to b$.

I will refer to condition (I) as the *classical implicative condition*, and I will call an *implicative lattice* any lattice which admits a binary operation satisfying (MP) and (I) $(=(\mathrm{I}^*))$. It can be shown that conditions (MP) and (I) uniquely specify the implication operation to be: $a \to b = \sup\{x \in L\colon a \wedge x \leqq b\}$. It can also be shown that every implicative lattice is distributive[12]. We also note that although not every distributive lattice is implicative, every complemented distributive (i.e., Boolean) lattice is, the implication operation being given by: $a \to b = a^\perp \vee b$. In other words, in the case of Boolean lattices, the implication operation is uniquely determined to be the classical material conditional.

Since the lattices of QM are not distributive, no implication satisfying both (MP) and (I) is definable on them; therefore, condition (I) is too strong a requirement in the case of QL. On the other hand, there is little reason to adopt condition (I) as a *minimal* criterion for implicationhood since in particular the strict implications of modal logic fail to satisfy it. Conditions (MP) and (E) are nevertheless too weak to specify in general a unique implication operation. In the case of general orthomodular lattices, considering only those binary operations which can be defined in terms of meet, join, and orthocomplement, we note that at least four distinct binary operations can be defined which satisfy the minimal implicative conditions (MP) and (E)[13]. These are four of the six generalized material conditionals investigated by Kotas[14].

(C1) $\quad C_1(a, b) = a^\perp \vee (a \wedge b)$,

(C2) $\quad C_2(a, b) = b \vee (a^\perp \wedge b^\perp)$,

(C3) $\quad C_3(a, b) = (a \wedge b) \vee (a^\perp \wedge b) \vee (a^\perp \wedge b^\perp)$,

(C4) $\quad C_4(a, b) = (a \wedge b) \vee (a^\perp \wedge b) \vee \vee \big((a^\perp \vee b) \wedge b^\perp\big)$.

From these initial investigations, we see that depending on how strenuous our criteria are, we can define on orthomodular lattices several distinct implication operations or no implication operations at all. It is therefore of interest to ascertain whether there is a set of conditions intermediate between the minimal implicative conditions, (MP) and (E), and the classical implicative conditions, (MP) and (I), which single out a unique implication for all orthomodular lattices. I have previously argued that operation C_1 is particularly appropriate as a conditional for QL[15]; also since it has received the greatest attention in the literature ([5, 6, 7, 16, 17]), we seek in particular conditions which single out C_1. The remainder of the paper will therefore be devoted to the examination of various attempts at a characterization of conditional C_1, which following Herman et al.[17] will be referred to occasionally as the "Sasaki hook".

3. The Finch Implicative Conditions

Following Finch[16], let us define a binary lattice operation, denoted ϱ, as follows.

(ϱ) $\quad a \varrho b = a \wedge (a^\perp \vee b)$.

It is convenient to regard ϱ as a "generalized meet operation"[13] in the sense that ϱ agrees with the ordinary meet operation for all compatible pairs.

(R) $\quad a \varrho b = a \wedge b$ if aKb.

We also note that ϱ is commutative if and only if the elements considered are compatible.

(C) $\quad a \varrho b = b \varrho a$ iff aKb.

Regarding this operation as a generalization of the meet operation, we can weaken the classical implicative conditions, defining the F-implicative conditions as follows.

(MP) $a \wedge (a \to b) \leqq b$,

(F) $a \varrho x \leqq b$ implies $x \leqq a \to b$.

We first note that condition (F) implies condition (E) and is implied by condition (I), whereas the converse implications do not generally hold. We also note that whenever a and x are compatible, $a \varrho x = a \wedge x$, and condition (F) reduces to condition (I). Let us call an ortholattice which admits a binary operation satisfying (MP) and (F) an *F-implicative lattice*. We then have the following theorems.

Theorem 1. Every orthomodular lattice is F-implicative.

Proof. Let $a \to b = C_1(a, b) = a^\perp \vee (a \wedge b)$. Then by the orthomodular law (OM3) $a \wedge (a^\perp \vee (a \wedge b)) \leqq b$; thus we have condition (MP). Now suppose $a \varrho x \leqq b$. Then $a \wedge (a^\perp \vee x) \leqq b$. Therefore,

$$a \wedge (a^\perp \vee x) \leqq a \wedge b\,,$$

and so,

$$a^\perp \vee (a \wedge (a^\perp \vee x)) \leqq a^\perp \vee (a \wedge b) = a \to b\,.$$

By F-H (noting $a^\perp K a$, $a^\perp K a^\perp \wedge x$),

$$a^\perp \vee (a \wedge (a^\perp \vee x)) = a^\perp \vee x\,.$$

But $x \leqq a^\perp \vee x$; therefore, $x \leqq a \to b$.

According to Theorem 1, in an orthomodular lattice, C_1 the Sasaki hook satisfies (MP) and (F). The following theorem asserts that only the Sasaki hook satisfies (MP) and (F).

Theorem 2. In an orthomodular lattice, there is exactly one operation — namely, C_1 — which satisfies conditions (MP) and (F).

Proof. According to the orthomodular law (OM3) $a \wedge (a^\perp \vee (a \wedge b)) \leqq b$, which is to say, $a \varrho (a \wedge b) \leqq b$. Therefore by (F), $a \wedge b \leqq a \to b$. Also $a \varrho a^\perp = a \wedge a^\perp \leqq b$; therefore by (F), $a^\perp \leqq a \to b$. Thus $a^\perp \vee (a \wedge b) \leqq a \to b$. According to (MP),

$$a \wedge (a \to b) \leqq b\,.$$

Consequently,

$$a \wedge (a \to b) \leqq a \wedge b\,,$$

and

$$a^\perp \vee (a \wedge (a \to b)) \leqq a^\perp \vee (a \wedge b)\,.$$

But from above, $a^\perp \leqq a \to b$; therefore by the orthomodular law (OM), $a^\perp \vee (a \wedge (a \to b)) = a \to b$. Thus $a \to b \leqq a^\perp \vee (a \wedge b)$. Finally we have $a \to b = a^\perp \vee (a \wedge b)$.

We note that the orthomodular law plays a crucial role in Theorem 2. An F-implicative lattice, however, is not necessarily orthomodular. On the other hand, the F-implicative conditions are not trivial conditions on ortholattices. This may be seen in the following theorems.

Theorem 3. Not every F-implicative lattice is orthomodular.

Proof. It is sufficient to produce an ortholattice which is F-implicative but not orthomodular. The following lattice L_6 is a well-known ortholattice which is not orthomodular.

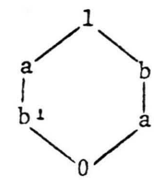

Fig. 1.

To see that L_6 is not orthomodular, we observe that $b^\perp \leqq a$ but $a \neq b^\perp \vee (a \wedge b) = b^\perp$. Thus L_6 does not satisfy (OM). We next observe that the operation given by the following table satisfies both (MP) and (F).

Table 1.

				y		
$x \to y$	1	a	b	$b^\perp$	$a^\perp$	0
1	1	a	b	$b^\perp$	$a^\perp$	0
a	1	1	b	b	b	$a^\perp$
x b	1	a	1	a	a	$b^\perp$
$a^\perp$	1	1	b	1	b	b
$b^\perp$	1	a	1	a	1	a
0	1	1	1	1	1	1

Theorem 4. Not every ortholattice is F-implicative.

Proof. We simply observe that the following ortholattice L_8 is not F-implicative.

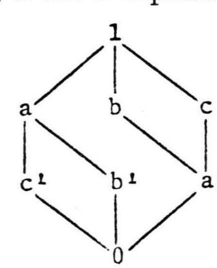

Fig. 2.

Consider whether an element $a \to c^\perp$ exists in L_8 which satisfies (MP) and (F). According to (F) $a \wedge (a^\perp \vee x) \leqq c^\perp$ implies $x \leqq a \to c^\perp$. In L_8 the antecedent is satisfied by $x=0$, $x=a^\perp$, $x=b$, and $x=c$. Therefore $1=0 \vee a^\perp \vee b \vee c \leqq b \to c^\perp$. Thus as required by condition (F), $a \to c^\perp = 1$. But this element does not satisfy (MP) since $a \wedge (a \to c^\perp) = a \wedge 1 = a$, but $a \nleqq c^\perp$. Thus no element $a \to c^\perp$ satisfying (MP) and (F) is definable on L_8. It is therefore not an F-implicative lattice.

Before closing this section, it is interesting to note that the generalized meet ϱ is related to the associated Sasaki hook $\to$ (operation C_1) in a way which is completely analogous to classical logic.

(A1) $\quad a \;\varrho\; b = (a \to b^\perp)^\perp$,

(A2) $\quad a \to b = (a \;\varrho\; b^\perp)^\perp$.

Besides (A1) and (A2), it can be shown that in any orthomodular lattice the Sasaki hook satisfies the following generalization of the classical implicative condition (I*). (See Sect. 5, Lemma 8.)

(F*) $\quad a \varrho x \leqq b$ iff $x \leqq a \to b$.

The Sasaki hook is therefore related to its associated generalized meet in a way completely analogous to the relation between the classical material implication and the ordinary meet. One might therefore ask whether any of the other Kotas connectives satisfy a similar set of laws. Specifically, one can define for each implication operation its associated generalized meet as follows.

(GM) $\quad M_i(a,b) = (C_i(a, b^\perp))^\perp$.

One then asks for each implication operation whether the following condition is satisfied in an orthomodular lattice.

(GI) $\quad M_i(a,x) \leqq b$ implies $x \leqq C_i(a,b)$.

We have already seen that operation C_1, the Sasaki hook, satisfies (GI). It can furthermore be shown that the Sasaki hook is the only Kotas connective which satisfies (GI)[13].

4. The Quasi-Implicative Conditions

We have seen that restrictions can be placed on the implication operation in such a way that a unique implication operation is singled out for QL. The proof of this result, however, depends on the orthomodular law. The question immediately arises as to whether one can begin with a general ortholattice, introduce an implication operation by means of conditions analogous to (MP) and (F) (alternatively (I)), and subsequently derive both the orthomodular law and the existence of a unique implication operation. Such a program has in fact been carried out by Mittelstaedt[7] who defines a *quasi-implicative lattice* to be an ortholattice admitting a binary operation, called the quasi-implication, satisfying the two following restrictions.

(MP) $\quad a \wedge (a \to b) \leqq b$,

(QI) $\quad a \wedge x \leqq b$ implies $a^\perp \vee (a \wedge x) \leqq a \to b$.

Mittelstaedt then proves that an ortholattice is quasi-implicative exactly if it is orthomodular, and that the quasi-implication is uniquely determined by restrictions (MP) and (QI) to be the Sasaki hook.

The difficulty with Mittelstaedt's quasi-implicative conditions is that restriction (QI) seems to lack the direct intuitive appeal of (E), (I), or (F). For example, it is not immediately clear from the form of (QI) what its precise connection with the classical implicative condition (I) is. One might therefore ask whether there are yet other sets of conditions which do the same work as (QI) (together with (MP)), but which have more intuitive plausibility as restrictions on the implication operation.

Accordingly, I intend to present three sets of restrictions, which I believe are more intuitively motivated, and show that each set is equivalent to the quasi-implicative conditions proposed by Mittelstaedt. The first set of conditions I wish to propose is a strengthening of the F-implicative conditions and are given as follows.

(MP) $\quad a \wedge (a \to b) \leqq b$,

(F) $\quad a \varrho x \leqq b$ implies $x \leqq a \to b$,

(H) $\quad a \wedge b \leqq a \to b$.

Restrictions (MP) and (F) comprise the F-implicative conditions. We also note that condition (H) is satisfied by the quasi-implication (Sasaki hook) in all orthomodular lattices. Therefore, in view of the results of the previous section concerning F-implicative lattices, it is sufficient for our purposes to prove that an F-implicative lattice which satisfies restriction (H) is orthomodular. Let us refer to such a lattice as an *H-implicative lattice*. We then have the following theorem.

Theorem 5. An H-implicative lattice is orthomodular.

Proof. $a \varrho a^\perp = a \wedge a^\perp \leqq b$. Therefore by (F), $a^\perp \leqq a \to b$. By (H) $a \wedge b \leqq \to b$. Therefore $a^\perp \vee (a \wedge b) \leqq a \to b$. By (MP) $a \wedge (a \to b) \leqq b$. Therefore,

$$a \wedge (a^\perp \vee (a \wedge b)) \leqq a \wedge (a \to b) \leqq b\,,$$

which is the orthomodular law (OM3).

Thus in view of theorems 1, 2, and 5, we can conclude that an H-implicative lattice is orthomodular and that the implication operation is uniquely determined to be the Sasaki hook or Mittelstaedt's quasi-implication.

Concerning the intuitive plausibility of the H-implicative conditions, we note that (F) is exactly like the classical implicative condition (I) except that $a \wedge x$ is replaced by $a \varrho x$. But whenever $a K x$, $a \wedge x = a \varrho x$; therefore, the only difference between condition (F) and condition (I) concerns those situations in which a and x are not compatible. It thus seems that (F) is a natural and straightforward generalization of the classical implicative condition (I). Concerning restriction (H), we note that it is a consequence of the classical implicative condition (I) and so is satisfied by both the classical material implication and the intuitionistic implication. It is also satisfied by counterfactual conditionals as well as by the four Kotas connectives C_1—C_4. Among the generally accepted implication connectives, only the strict implications of modal logic fail to satisfy (H). Thus condition (H) appears to be a plausible restriction on a *non-strict* implication.

The intuitive connection between the H-implicative conditions and the classical implicative conditions can furthermore be seen as follows. We can strengthen the H-implicative conditions by replacing condition (H) by the following stronger condition.

(H*) $\qquad b \leqq a \to b\,.$

Let us call an *H*-implicative lattice* any ortholattice admitting a binary operation satisfying conditions (MP), (F), and (H*). We then have the following theorem.

Theorem 6. An ortholattice is H*-implicative if and only if it is implicative.

Proof. To prove that every implicative ortholattice is H*-implicative it is sufficient to show that (F) and (H*) follow from (MP) and (I). Concerning (F), suppose $a \varrho x \leqq b$. Then since $a \wedge x \leqq a \varrho x$, $a \wedge x \leqq b$. Therefore by (I), $x \leqq a \to b$. Concerning (H*), we note that $a \wedge b \leqq b$. Therefore by (I), $b \leqq a \to b$. Thus an implicative ortholattice is H*-implicative. Conversely, suppose an ortholattice is H*-implicative. Then by (F), $a^\perp \leqq a \to b$, and by (H) $b \leqq a \to b$. Therefore $a^\perp \vee b \leqq a \to b$. By (MP) $a \wedge (a \to b) \leqq b$. Therefore $a \wedge (a^\perp \vee b) \leqq b$, and so $a \wedge (a^\perp \vee b) \leqq a \wedge b$. Now suppose $a \wedge x \leqq b$. Then since $a \wedge (a^\perp \vee x) \leqq a \wedge x$, $a \wedge (a^\perp \vee x) \leqq b$. Therefore by (F), $x \leqq a \to b$. We thus have condition (I), and since condition (MP) is immediate we are through.

Having shown that the H*-implicative conditions are equivalent to the classical implicative conditions for the class of ortholattices, we see how the H-implicative conditions, which are equivalent to Mittelstaedt's quasi-implicative conditions, are a straightforward generalization of the classical implicative conditions (H*-implicative conditions) in which restriction (H*) is replaced by the weaker (and more plausible!) restriction (H).

We now consider a second set of conditions, equivalent to Mittelstaedt's quasi-implicative conditions, introduced on the basis of the following motivation. The apparent reason that the quasi-implication (Sasaki hook) of orthomodular lattices and QL fails to satisfy the classical implicative condition (I) is that the elements a and x might fail to be compatible. We have already observed that whenever $a K x$ the F-implicative condition (F) reduces to the classical implicative condition (I). We also have the following corollary to Mittelstaedt's condition (QI).

Corollary to (QI). $a \wedge x \leqq b$ implies $x \leqq a \to b$, if $a K x$.

Proof. Suppose $a K x$ and $a \wedge x \leqq b$. Then by (QI), $a^\perp \vee (a \wedge x) \leqq a \to b$. By supposition, $a K x$, and always $a K a^\perp$. Therefore by the Foulis-Holland theorem, $a^\perp \vee (a \wedge x) = a^\perp \vee x$. But $x \leqq a^\perp \vee x$; therefore, $x \leqq a \to b$.

Recall the definition of the relation of compatibility among ortholattice elements.

(K) $\qquad a K b \quad \text{iff} \quad a = (a \wedge b) \vee (a \wedge b^\perp)\,.$

Now we can use (K) as a means of generalizing the classical implicative condition (I). Specifically we can substitute for x the expression $(x \wedge a) \vee (x \wedge a^\perp)$ which equals x whenever x and a are compatible. As a result of this substitution, on the left hand side of (I) we have $a \wedge ((x \wedge a) \vee (x \wedge a^{\perp}))$ which *in general* equals $a \wedge x$. On the right hand side of (I) we have $(x \wedge a) \vee (x \wedge a^{\perp}) \leqq a \rightarrow b$. We thus arrive that the following condition called the Q-implicative condition.

(Q) $\quad a \wedge x \leqq b$ implies $(x \wedge a) \vee (x \wedge a^{\perp}) \leqq a \rightarrow b$.

The intuitive plausibility of condition (Q) is that it is exactly like condition (I) except that the expression x is replaced by the expression $(x \wedge a) \vee (x \wedge a^{\perp})$ which equals x whenever a and x are compatible. Its connection with the classical implicative condition is then fairly explicit. It is a straightforward generalization of the classical implicative condition (I) which takes into consideration the possibility that the elements a and x are not compatible. We next prove several theorems concerning the Q-implicative conditions. Let us call a *Q-implicative lattice* any ortholattice admitting a binary operation satisfying conditions (MP) and (Q). We then have the following theorems.

Theorem 7. Every orthomodular lattice is Q-implicative.

Proof. Let $a \rightarrow b = a^{\perp} \vee (a \wedge b)$. Then by the orthomodular law (OM3), $a \wedge (a^{\perp} \vee (a \wedge b)) \leqq b$, so we have condition (MP). Concerning condition (Q), suppose $a \wedge x \leqq b$. Then $a \wedge x \leqq a \wedge b$, and $a^{\perp} \vee (a \wedge x)$ $\leqq a^{\perp} \vee (a \wedge b) = a \rightarrow b$. But

$$(x \wedge a) \vee (x \wedge a^{\perp}) \leqq a^{\perp} \vee (a \wedge x) .$$

Thus $(x \wedge a) \vee (x \wedge a^{\perp}) \leqq a \rightarrow b$.

Theorem 8. Every Q-implicative lattice is orthomodular.

Proof. Since $a \wedge b \leqq b$, by (Q) we have

$$(b \wedge a) \vee (b \wedge a^{\perp}) \leqq a \rightarrow b .$$

Also since $a \wedge a^{\perp} \leqq b$, by (Q) we have

$$a^{\perp} = (a^{\perp} \wedge a) \vee (a^{\perp} \wedge a^{\perp}) \leqq a \rightarrow b .$$

Therefore,

$$a^{\perp} \vee (b \wedge a) \vee (b \wedge a^{\perp}) = a^{\perp} \vee (a \wedge b) \leqq a \rightarrow b .$$

But by condition (MP), $a \wedge (a \rightarrow b) \leqq b$. Therefore, $a \wedge (a^{\perp} \vee (a \wedge b)) \leqq b$ which is the orthomodular law (OM3).

Theorem 9. In a Q-implicative lattice, the implication operation is uniquely determined to be: $a \rightarrow b = a^{\perp} \vee (a \wedge b)$.

Proof. From Theorem 8, we already have $a^{\perp} \vee (a \wedge b) \leqq a \rightarrow b$. We now prove the converse relation. By condition (MP) $a \wedge (a \rightarrow b) \leqq b$. Consequently, $a \wedge (a \rightarrow b) \leqq a \wedge b$, and $a^{\perp} \vee (a \rightarrow b)) \leqq a^{\perp} \vee (a \wedge b)$. From Theorem 8, we have $a^{\perp} \leqq a \rightarrow b$. Therefore by the orthomodular law (OM)

$$a^{\perp} \vee (a \wedge (a \rightarrow b)) = a \rightarrow b .$$

Thus $a \rightarrow b \leqq a^{\perp} \vee (a \wedge b)$, and finally

$$a \rightarrow b = a^{\perp} \vee (a \wedge b) .$$

We finally consider a third set of conditions which are equivalent to Mittelstaedt's quasi-implicative conditions. We first note that in the case of the strict implications of modal system S4, there is a special S4 law of exportation analogous to the classical implicative condition (I).

(S4) $\quad a \wedge x \leqq b$ implies $x \leqq a \rightarrow b$
for all strict implicative x.

An element x is strict implicative exactly if $x = a \rightarrow b$ for some a, b. Of course if every lattice element is strict implicative, then the S4 law of exportation reduces to the classical implicative condition (I), and the strict implication reduces to the classical implication. However, for non-trivial modal systems, not every element is strict implicative.

In view of the S4 law of implication, we might ask whether there is a similar law for orthomodular lattices and QL, which resembles the classical implicative condition (I) except for a restriction on the character of the element x. We have already seen that condition (F), which is satisfied by orthomodular lattices, reduces to condition (I) whenever a and x are compatible. The obvious candidate is then the following.

(K1) $\quad a \wedge x \leqq b$ implies $x \leqq a \rightarrow b$
for all x compatible with a.

The difficulty with this condition is that the relation of compatibility as defined by (K) is not symmetric unless the lattice in question is orthomodular. But since we wish to derive orthomodularity, not assume it from the outset, we cannot assume that "compatible" in Eq. (K1) is symmetric. More specifically, we must decide which sense of "compatible" is meant in (K1), xKa or aKx. I propose to employ 'xKa' in place of 'x compatible with a'. (K1) then becomes the following.

(K2) $a \wedge x \leqq b$ implies $x \leqq a \to b$ for all x such that xKa.

Let us call a *K-implicative lattice* any ortholattice admitting a binary operation satisfying condition (MP) and condition (K2). We then have the following theorems.

Theorem 10. Every orthomodular lattice is K-implicative.

Proof. Let $a \to b = a^\perp \vee (a \wedge b)$. We recall Theorem 7 which asserts that every orthomodular lattice is Q-implicative. As in Theorem 7, condition (MP) is satisfied. Now suppose xKa and $a \wedge x \leqq b$. Then by condition (Q), $(x \wedge a) \vee (x \wedge a^\perp) \leqq a \to b$. But by supposition xKa; therefore $x = (x \wedge a) \vee (x \wedge a^\perp)$. Thus $x \leqq a \to b$.

Theorem 11. Every K-implicative lattice is orthomodular.

Proof. $a^\perp = (a^\perp \wedge a) \vee (a^\perp \wedge a^\perp)$; thus $a^\perp K a$. Therefore, since $a \wedge a^\perp \leqq b$, by (K2) $a^\perp \leqq a \to b$. Also $a \wedge b = ((a \wedge b) \wedge a) \vee ((a \wedge b) \wedge a^\perp))$; thus $a \wedge b K a$. Therefore, since $a \wedge (a \wedge b) \leqq b$, by (K2) $a \wedge b \leqq a \to b$. Consequently $a^\perp \vee (a \wedge b) \leqq a \to b$. But by condition (MP), $a \wedge (a \to b) \leqq b$. Therefore $a \wedge (a^\perp \vee (a \wedge b)) \leqq b$, which is the orthomodular law (OM3).

Theorem 12. In a K-implicative lattice, the implication operation is uniquely determined to be: $a \to b = a^\perp \vee (a \wedge b)$.

Proof. The proof is similar to the proof of Theorem 9.

5. Properties of the Quasi-Implication

Having provided a number of characterizations of quasi-implicative lattices, in this Sect. I wish to investigate various properties of the quasi-implication (Sasaki hook) as defined on the general class of orthomodular lattices. Let us first recall some features of the calculus of classical propositional logic (CPL). The language of CPL may be formulated using an infinite list of propositional variables ($p, q, r, \ldots$), a binary connective $\to$ and a nullary connective f, together with signs for grouping and the usual rules of formula formation. The axiom schemes of CPL may be given as follows. A, B, C are metalinguistic variables ranging over formulas of the language of CPL.

(A1) $A \to (B \to A)$,

(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(A3) $((A \to B) \to A) \to A$,

(A4) $f \to A$.

The rule of transformation is modus ponens: from A and $A \to B$, infer B. The remaining logical signs representing negation, disjunction, and conjunction, are introduced as abbreviations according to the following schemes.

(N) $\neg A =: A \to f$,

(or) A or $B =: (A \to B) \to B$,

(&) A & $B =: (A \to (B \to f)) \to f$.

I use '=:' to mean "is an abbreviation for". Using these three abbreviation schemes, formulas in which defined connectives appear can be converted into formulas in purely primitive notation.

The four axioms of CPL, as well as all theorems of CPL, are *Boolean-valid*. This is to say if we convert any given axiom or theorem scheme into a Boolean lattice polynomial, replacing the metalinguistic variables by ordinary variables, and replacing $\to$ by the Boolean lattice implication operation and f by 0, then the resulting polynomial will be identically equal to the unit 1 for all Boolean lattices. For example, (A1) is Boolean valid because $a \to (b \to a) = 1$ for all a, b in all Boolean lattices, where $a \to b = a^\perp \vee b$. Now suppose instead we convert (A1)—(A4) into orthomodular lattice polynomials, replacing $\to$ by the quasi-implication and f by 0. Then which of the resulting lattice polynomials will be identically equal to the unit for all orthomodular lattices; in other words, which of these axioms are *orthomodular-valid* (OM-valid), where $\to$ is understood as quasi-implication. As it turns out, only (A3) and (A4) are OM-valid. The OM-validity of (A3), known as Peirce's law, has been shown by Mittelstaedt[7]; the OM-validity of (A4) is an easy verification: $0 \to a = 1$ for all a since $0 \leqq a$ for all a. On the other hand, (A1) is invalidated by orthomodular lattice L_6.

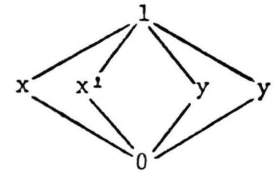

Fig. 3.

Substitute $x|a$ and $y|b$. Then $b \to a = y \to x = y^{\perp}$, and $a \to (b \to a) = x \to y^{\perp} = x^{\perp} \neq 1$.

A2) is invalidated by the lattice L(H3) of subspaces of a three dimensional Hilbert space H3. As shown in a previous paper[15], in the lattice of subspaces of a Hilbert space, the quasi-implication can be characterized as follows. [x is any vector; A, B, C are projections associated with the subspaces $|A|$, $|B|$, $|C|$. $\to$ is the quasi-implication: $A \to B = A^{\perp} \vee (A \wedge B) = A^{\perp} + (A \wedge B)$; $|A \to B| = |A|^{\perp} \oplus (|A| \cap |B|)$.]

(HS) $\quad x \in |A \to B|$ iff $Ax \in |B|$.

The question of whether (A2) is valid in L(H3) then reduces to the question of whether the following holds for all projections A, B, C on H3.

(S) $\quad |A \to (B \to C)| \subseteqq |(A \to B) \to (A \to C)|$.

By (HS) $x \in |A \to (B \to C)|$ iff $Ax \in |B \to C|$ iff $BAx \in |C|$. Also, $x \in |(A \to B) \to (A \to C)|$ iff $(A \to B)x \in |A \to C|$ iff $A(A \to B)x \in |C|$. But $A(A \to B) = A(A^{\perp} + (A \wedge B)) = AA^{\perp} + A(A \wedge B) = (A \wedge B)$. Thus $x \in |(A \to B) \to (A \to C)|$ iff $(A \wedge B)x \in |C|$. Now the image BA(H3) of the vector space H3 under the operator BA is given by: $BA(\mathrm{H3}) = |B| \cap (|B|^{\perp} \oplus |A|)$. The image of H3 under $A \wedge B$ is $|A \wedge B| = |A| \cap |B|$, which is a proper subset of BA(H3) unless A and B are compatible (commute). Therefore, whenever A and B are not compatible there will be vectors x in H3 such that $BAx \notin |A| \cap |B|$. Furthermore if $|A|$ and $|B|$ are two-dimensional subspaces, then there will be vectors among these which are not orthogonal to $|A| \cap |B|$. Take one such vector, call it w. Then let $|C|$ be the subspace generated by BAw, that is, the set of scalar multiples of BAw. Clearly $BAw \in |C|$; thus $w \in |A \to (B \to C)|$. On the other hand, $(A \wedge B)w \notin |C|$. Suppose otherwise; then $(A \wedge B)w = cBAw$ for some scalar $c \neq 0$ (Recall non-orthogonality.); therefore, since $(A \wedge B)w \in |A \wedge B| = |A| \cap |B|$, it follows that $BAw \in |A| \cap |B|$, which is contrary to our original supposition. Thus $w \notin |(A \to B) \to (A \to C)|$, and consequently $|A \to (B \to C)|$ is not a subset of $|(A \to B) \to (A \to C)|$.

Having shown that the classical axioms (A1) and (A2) are not OM-valid, we now consider various weakened versions of these laws which are satisfied by all orthomodular lattices. We first prove a simple lemma stating that the quasi-implication $a \to b$ reduces to the classical conditional whenever and only whenever aKb.

Lemma 1. $a \to b = a^{\perp} \vee b$ iff aKb.

Proof. Suppose $a \to b = a^{\perp} \vee b$. Then $a \wedge (a^{\perp} \vee b) = a \wedge (a \to b) = a \wedge (a^{\perp} \vee (a \wedge b)) =$ (by F-H) $= a \wedge b$. Therefore, $(a \wedge b) \vee (a \wedge b^{\perp}) = (a \wedge (a^{\perp} \vee b)) \vee (a \wedge b^{\perp})$ $=$ (by F-H) $= a$. Thus aKb. Conversely, suppose aKb. Then $a \to b = a^{\perp} \vee (a \wedge b) =$ (by F-H) $= a^{\perp} \vee b$.

We next prove that axiom (A1) is satisfied by an orthomodular lattice exactly if the elements considered are compatible.

Theorem 13. $a \to (b \to a) = 1$ iff aKb.

Proof. Suppose $a \to (b \to a) = 1$. Then $a \leqq b \to a$, and so $a \leqq a \wedge (b \to a)$. But

$$a \wedge (b \to a) = a \wedge (b^{\perp} \vee (b \wedge a)) = (\text{by F-H}) = (a \wedge b^{\perp}) \vee (a \wedge b)\,.$$

Since in general $(a \wedge b) \vee (a \wedge b^{\perp}) \leqq a$, it follows that $a = (a \wedge b) \vee (a \wedge b^{\perp})$, and aKb. Conversely suppose aKb. Then by Lemma 1, $b \to a = b^{\perp} \vee a$. But $a \leqq b^{\perp} \vee a$; therefore $a \leqq b \to a$, and $a \to (b \to a) = 1$.

As an immediate corollary to Theorem 13, we have the converse of Peirce's law (A3).

Corollary to Theorem 13. $a \to ((a \to b) \to a) = 1$

Proof. Since $aK(a \to b)$, by Theorem 13

$$a \to ((a \to b) \to a) = 1\,.$$

Before examining a weakened version of axiom (A2), we prove two lemmas.

Lemma 2. $a \to (b \to c) = (a \wedge b) \to c$ iff aKb.

Proof. Suppose $a \to (b \to c) = (a \wedge b) \to c$. Then in particular $a \to (b \to a) = (a \wedge b) \to a = 1$. Thus by Theorem 13, aKb. Conversely, suppose aKb. Then

$$\begin{aligned} a \to (b \to c) &= a^{\perp} \vee [a \wedge (b^{\perp} \vee (b \wedge c))] \\ &= (\text{by F-H}) = a^{\perp} \vee (a \wedge b) \vee (a \wedge b \wedge c) \\ &= (\text{by F-H}) = a^{\perp} \vee b^{\perp} \vee (a \wedge b \wedge c) \\ &= (a \wedge b) \to c\,. \end{aligned}$$

Lemma 3. $(a \to b) \to (a \to c) = (a \wedge b) \to c$.

Proof. $aK(a \to b)$. Therefore by Lemma 2, $(a \to b) \to (a \to c) = ((a \to b) \wedge a) \to c$. But $(a \to b) \wedge a$ $= a \wedge (a^{\perp} \vee (a \wedge b)) =$ (by F-H) $= a \wedge b$. Thus

$$(a \to b) \to (a \to c) = (a \wedge b) \to c\,.$$

The following theorem provides various conditions under which (A2) is satisfied by an orthomodular lattice.

Theorem 14. $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$ if any of the following conditions obtain: a) aKb, b) $aK(b \wedge c)$, c) $cK(a \wedge b)$.

Proof. $a \leqq b$ iff $a \to b = 1$, and by Lemma 3 $(a \to b) \to (a \to c) = (a \wedge b) \to c$. It is therefore sufficient in each case to prove $a \to (b \to c) \leqq (a \wedge b) \to c$.

a) Immediate from Lemma 2.

b) Suppose $aK(b \wedge c)$. Then

$$\begin{aligned} a \to (b \to c) &= a^\perp \vee [a \wedge (b^\perp \vee (b \wedge c))] \\ &= (\text{by F-H}) = a^\perp \vee (a \wedge b^\perp) \vee (a \wedge b \wedge c) \\ &\leqq a^\perp \vee b^\perp \vee (a \wedge b \wedge c) = (a \wedge b) \to c. \end{aligned}$$

c) Suppose $cK(a \wedge b)$. Then by Lemma 1,

$$(a \wedge b) \to c = (a \wedge b)^\perp \vee c = a^\perp \vee b^\perp \vee c.$$

But $a \to (b \to c) = a^\perp \vee [a \wedge (b^\perp \vee (b \wedge c))] \leqq a^\perp \vee b^\perp \vee c$. Thus $a \to (b \to c) \leqq (a \wedge b) \to c$.

The following lemma is a consequence of the minimal implicative condition (MP) and is therefore a property of all implication operations insofar as they satisfy the minimal implicative conditions. It is the *law of importation*; note that the converse law of exportation is the classical implicative condition (I).

Lemma 4. $a \leqq b \to c$ implies $a \wedge b \leqq c$.

Proof. Suppose $a \leqq b \to c$. Then $a \wedge b \leqq b \wedge (b \to c)$. But by (MP) $b \wedge (b \to c) \leqq c$. Thus $a \wedge b \leqq c$.

The following theorem provides a weakened version of (A2) satisfied by the quasi-implication.

Theorem 15. $a \to (b \to c) = 1$ implies $(a \to b) \to (a \to c) = 1$.

Proof. Suppose $a \to (b \to c) = 1$. Then $a \leqq b \to c$. Therefore by Lemma 4, $a \wedge b \leqq c$, and so $(a \wedge b) \to c = 1$. Finally by Lemma 3, $(a \to b) \to (a \to c) = 1$.

We now consider the transitivity properties of the quasi-implication. We first note that this operation satisfies a weak transitive law satisfied by all implication operations.

(WT) $\quad a \to b = 1$ and $b \to c = 1$ implies $a \to c = 1$.

(WT) is an immediate consequence of the transitivity of the partial order relation and the fact that $a \leqq b$ iff $a \to b = 1$, which is true of every binary lattice operation satisfying the minimal implicative conditions. However, (WT) is a very weak sort of transitive law. In classical logic, the strong form of the transitive law is usually given in two forms, expressed lattice theoretically as follows:

(T1) $\quad (a \to b) \leqq (b \to c) \to (a \to c)$,

(T2) $\quad (b \to c) \leqq (a \to b) \to (a \to c)$.

Since $1 \leqq a$ only if $a = 1$, it is easy to see that (WT) is an immediate consequence of either (T1) or (T2). Another consequence of (T1) or (T2) may be obtained by applying the law of importation (Lemma 4).

(T3) $\quad (a \to b) \wedge (b \to c) \leqq (a \to c)$.

As is easily verified, (WT) is also an immediate consequence of (T3). Another consequence of (T3) is the classical law of "weakening", also known as the law of "strengthening the antecedent".

(W) $\quad b \to c \leqq (a \wedge b) \to c$.

(W) may be obtained from (T3) by substituting $a \wedge b$ for a and noting that $(a \wedge b) \to b = 1$. It can moreover be shown that any implication operation satisfying (W) as well as the "identity law": $1 \to a = a$ for all a, also satisfies the classical axiom (A1). We simply substitute 1 for b in (W), which thereby becomes $1 \to c \leqq (a \wedge 1) \to c$ which is equivalent to: $c \leqq a \to c$. Now the quasi-implication does satisfy the identity law, but it does not in general satisfy (A1) (which was invalidated by lattice L_6). It then follows that the quasi-implication does not satisfy (W), and hence it does not satisfy any of the transitive laws (T1)—(T3).

Having shown that the quasi-implication does not in general satisfy the strong forms of the transitive law, we shall presently consider various weakened version of these laws which it does satisfy. However, first we define the *quasi-equivalence* or *quasi-biconditional* associated with the quasi-implication. We then prove that it has a form similar to the classical biconditional.

Definition 1. $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

Lemma 5. $a \leftrightarrow b = (a \wedge b) \vee (a^\perp \wedge b^\perp)$.

Proof.
$$\begin{aligned} a \leftrightarrow b &= (a \to b) \wedge (b \to a) \\ &= (a^\perp \vee (a \wedge b)) \wedge (b^\perp \vee (b \wedge a)) \\ &= (\text{by F-H}) \\ &= [(a^\perp \vee (a \wedge b)) \wedge b^\perp] \vee (a \wedge b) \\ &= (\text{by F-H}) = (a^\perp \wedge b^\perp) \vee (a \wedge b). \end{aligned}$$

The following theorem expresses a weakened version of (T3) satisfied by the quasi-implication.

Theorem 16.

(WT3) $(b \to a) \wedge (a \to b) \wedge (b \to c) \leqq (a \to c)$.

Proof. $(b \to a) \wedge (a \to b) \wedge (b \to c) =$ (by Def. 1)
$= (a \leftrightarrow b) \wedge (b \to c) =$ (by Lem. 5)
$= [(a \wedge b) \vee (a^\perp \wedge b^\perp)] \wedge [b^\perp \vee (b \wedge c)] =$ (by F-H)
$= [((a \wedge b) \vee (a^\perp \wedge b^\perp)) \wedge b^\perp]$
$\vee [((a \wedge b) \vee (a^\perp \wedge b^\perp)) \wedge (b \wedge c)] =$ (by F-H)
$= [a^\perp \wedge b^\perp] \vee [a \wedge b \wedge c] \leqq a^\perp \vee (a \wedge c) = a \to c$.

The next theorem, which is a corollary to Theorem 16, states that the quasi-biconditional satisfies the biconditional transitive law.

Theorem 17. $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leqq (a \leftrightarrow c)$.

Proof. By Def. 1,

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) = (a \to b) \wedge (b \to a) \wedge (b \to c) \wedge (c \to b).$$

Thus

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leqq (a \to b) \wedge (b \to a) \wedge (b \to c),$$

and

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leqq (b \to a) \wedge (b \to c) \wedge (c \to b).$$

But by Theorem 16,

$$(a \to b) \wedge (b \to a) \wedge (b \to c) \leqq (a \to c),$$

and

$$(b \to a) \wedge (b \to c) \wedge (c \to b) \leqq (c \to a).$$

Thus

$$(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leqq (a \to c) \wedge (c \to a) = (a \leftrightarrow c).$$

Recall that the law of weakening (W) may be obtained from the transitive law (T3) by substituting $a \wedge b$ for a in (T3). We can perform the same substitution into (WT3) and thereby obtain a weakened version of (W) satisfied by the quasi-implication. However, first we prove four relevant lemmas.

Lemma 6, $a \to (a \wedge b) = a \to b$.

Proof. $a \to (a \wedge b) = a^\perp \vee (a \wedge (a \wedge b)) = a^\perp \vee (a \wedge b) = a \to b$.

Lemma 7. $a \leqq b$ iff $x \leqq a$ implies $x \leqq b$ for all x.

Proof. Suppose $a \leqq b$ and $x \leqq a$. Then by transitivity, $x \leqq b$. Suppose $x \leqq a$ implies $x \leqq b$ for all x. Then in particular $a \leqq a$ implies $a \leqq b$. Thus $a \leqq b$.

The following lemma states that the quasi-implication satisfies an importation-exportation law completely analogous to the one satisfied by the classical implication. [See (I*) in Sect. 2.]

Lemma 8. $a \circ x \leqq b$ iff $x \leqq a \to b$.

Proof. Suppose $x \leqq a \to b$. Then

$$a^\perp \vee x \leqq a^\perp \vee (a \to b),$$

and

$a \wedge (a^\perp \vee x) \leqq a \wedge (a^\perp \vee (a \to b))$.
But $aK(a \to b)$, so by F-H,
$a \wedge (a^\perp \vee (a \to b)) = a \wedge (a \to b)$.
But by (MP), $a \wedge (a \to b) \leqq b$. Thus

$$a \circ x = a \wedge (a^\perp \vee x) \leqq b.$$

The converse has already been proved in Theorem 1.

The following lemma shows that the quasi-implication satisfies a law satisfied by all generally accepted implication connectives.

Lemma 9. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Proof. Suppose $x \leqq a \to (b \wedge c)$. Then by Lemma 8, $a \circ x \leqq (b \wedge c)$. Therefore $a \circ x \leqq b$ and $a \circ x \leqq c$. Then by Lemma 8, $x \leqq a \to b$ and $x \leqq a \to c$. Consequently $x \leqq (a \to b) \wedge (a \to c)$. Since each of the steps in this derivation is reversible, we have $x \leqq a \to (b \wedge c)$ iff $x \leqq (a \to b) \wedge (a \to c)$. Thus by Lemma 7, $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

We are now in a position to prove a weakened version of (W).

Theorem 18. $b \to (a \wedge c) \leqq (a \wedge b) \to c$.

Proof. Substituting $a \wedge b$ for a in (WT3), we obtain $(b \to (a \wedge b)) \wedge ((a \wedge b) \to a) \wedge (b \to c) \leqq (a \wedge b) \to c$. Now $(a \wedge b) \to a = 1$, and by Lemma 6, $b \to (b \wedge a) = b \to a$. We therefore obtain

$$(b \to a) \wedge (b \to c) \leqq (a \wedge b) \to c.$$

Thus by Lemma 9, $b \to (a \wedge c) \leqq (a \wedge b) \to c$.

We now prove that the quasi-implication satisfies a law which is a weakened version of the transitive law (T2) as well as of classical axiom (A2).

Theorem 19.

(WT2) $(b \to a) \to (b \to c) \leqq (a \to b) \to (a \to c)$.

Proof. By Lemma 3, $(b \to a) \to (b \to c) = (b \wedge a) \to c$, and $(a \to b) \to (a \to c) = (a \wedge b) \to c$. But $(b \wedge a) \to c = (a \wedge b) \to c$.

As an immediate consequence of (WT3), we have the following weakened version of the transitive law (T2).

Theorem 20.

(WT2′) $b \to c = 1$ implies $(a \to b) \to (a \to c) = 1$.

Proof. Suppose $b \to c = 1$. Then $(b \to a) \to (b \to c) = 1$. Therefore by (WT2), $(a \to b) \to (a \to c) = 1$.

Turning to the transitive law (T1), we note that the weakened version of (T1) analogous to (WT2′) does not obtain in all orthomodular lattices. In other words, the following is *not* a theorem of orthomodular lattice theory.

(T1 ?) $a \to b = 1$ implies $(b \to c) \to (a \to c) = 1$.

(T1 ?) is invalidated by lattice L_{12} which is reducible to the direct product of lattice L_6 (Fig. 3) and the two element Boolean lattice.

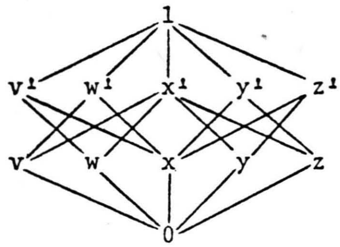

Fig. 4.

Consider the substitution: $y \mid a$, $x^\perp \mid b$, $w \mid c$. Then $a \to b = y \to x^\perp = 1$. But

$$\begin{aligned}(b \to c) \to (a \to c) &= (x^\perp \to w) \to (y \to w)\\ &= v^\perp \to y^\perp = w^\perp \neq 1.\end{aligned}$$

We finally consider two weakened versions of the transitive laws (T1) and (T2) given in the following theorem.

Theorem 21. In an orthomodular lattice, the quasi-implication satisfies the following weakened versions of (T 1) and (T 2).

(T1w) If $b \leqq a$, then $(a \to b) \leqq (b \to c) \to (a \to c)$,

(T2w) If $b \leqq a$, then $(b \to c) \leqq (a \to b) \to (a \to c)$.

Proof. (T1w). Suppose $b \leqq a$. Then aKb, and by Lemma 1 $a \to b = a^\perp \vee b$. Also,

$$(b \to c) \to (a \to c) = (b \to c)^\perp \vee ((b \to c) \wedge (a \to c)).$$

But

$$\begin{aligned}&(b \to c) \wedge (a \to c)\\ &\quad = (b^\perp \vee (b \wedge c)) \wedge (a^\perp \vee (a \wedge c)) = (\text{by F-H})\\ &\quad = [(b^\perp \vee (b \wedge c)) \wedge a^\perp] \vee [(b^\perp \vee (b \wedge c)) \wedge (a \wedge c)]\\ &\quad = (\text{since } a^\perp \leqq b^\perp) = a^\perp \vee [(b^\perp \vee (b \wedge c)) \wedge (a \wedge c)]\\ &\quad = (\text{by F-H}) = a^\perp \vee (b^\perp \wedge a \wedge c) \vee (b \wedge c).\end{aligned}$$

Thus

$$\begin{aligned}&(b \to c) \to (a \to c)\\ &\quad = (b \wedge (b^\perp \vee c^\perp)) \vee a^\perp \vee (b^\perp \wedge a \wedge c) \vee (b \wedge c).\end{aligned}$$

But

$$(b \wedge (b^\perp \vee c^\perp)) \vee (b \wedge c) = (\text{by F-H}) = b.$$

Thus

$$(b \to c) \to (a \to c) = b \vee a^\perp \vee (b^\perp \wedge a \wedge c).$$

But

$$a \to b = a^\perp \vee b \leqq b \vee a^\perp \vee (b^\perp \wedge a \wedge c).$$

It follows that $(a \to b) \leqq (b \to c) \to (a \to c)$.

(T2w). Suppose $b \leqq a$. Then $b \to a = 1$, and so

$$(b \to a) \to (b \to c) = (b \to c).$$

But by (WT2),

$$(b \to a) \to (b \to c) \leqq (a \to b) \to (a \to c).$$

Thus $(b \to c) \leqq (a \to b) \to (a \to c)$.

6. Conclusion

We have seen that a number of distinct binary lattice operations can be defined on general orthomodular lattices which satisfy the minimal implicative conditions (MP) and (E). We have also seen that (E) can strengthened in a variety of ways so as to select a unique implication operation, which we have variously called the Sasaki hook and the quasi-implication. We have furthermore observed that the quasi-implication fails to satisfy a number of laws associated with the classical material conditional, including the law of weakening and the strong transitive laws, although it was shown to satisfy a number of weakened versions of these laws.

Jauch and Piron[4] have argued that quantum logic admits no reasonable material implication and conclude that it is "very questionable whether we may properly call the lattice of general quantum mechanics a logic". What counts as a reasonable or acceptable implication is perhaps debateable. I have proposed that any binary operation satisfying criteria (MP) and (E) is a minimally acceptable implication, and I have shown that at least four distinct binary orthomodular lattice operations satisfy both these criteria. What additional criteria one chooses to impose on an implication operation is probably a matter of taste. For example, many readers will doubtless regard the failure of an operation, such as the quasi-implication, to satisfy the transitive law (T3) as a strong argument against admitting that operation as an acceptable implication. However, I cannot admit (T3) as a *minimal* criterion for implicationhood because this would eliminate counterfactual conditionals as legitimate implication connectives.

In a previous paper[15], I have argued that, within the framework of the lattice of subspaces of Hilbert space, the quasi-implication can be interpreted as a Stalnaker (counterfactual) conditional. It is generally agreed that a counterfactual conditional should not satisfy a number of laws associated with the classical material conditional, most particularly the law of weakening, and hence the strong laws of transitivity. For example, from the assertion, "If I *were* to drop this glass onto the floor, then it *would* break." one cannot validly infer the assertion, "If I *were* to drop this glass onto the floor *and* the floor *were* covered with foam, then it *would* break." It might be noted however that, like the quasi-implication, the Stalnaker conditional does in general satisfy the weakened transitive law (WT3).

In view of the findings of this paper and previous papers ([6,7,13,14,15,17]), I feel confident in claiming that the above mentioned objection of Jauch and Piron has been answered. Quantum logic does admit a reasonable implication connective. Indeed the implication connective investigated in this paper can be characterized from two entirely independent viewpoints. The quantum implication can first of all be characterized within the abstract framework of quasi-implicative lattices (orthomodular lattices), as we have done in this paper. It can also be characterized somewhat more concretely within the framework of the lattice of subspaces of Hilbert space and the theory of Stalnaker conditionals. That we have answered the objection of Jauch and Piron is not, however, to say that all objections to the word 'logic' in 'quantum logic' have been answered or can be answered.*

* *Note added in proof:* Conditions (MP) and (E) are minimal criteria for implicationshood in the sense that they should be satisfied by any implication operation on any logic (lattice) regardless of whether negation (orthocomplementation) is defined. Besides the minimal implicative conditions which should hold for any logic, with or without negation, there is an additional criterion which presents itself as plausible for any logic with negation (lattice with orthocomplementation): the law of modus tollens. According to this law $P \rightarrow Q$ together with the negation of Q should entail the negation of P. Lattice theoretically this is expressed as follows: $b^{\perp} \wedge (a \rightarrow b) \leqq a^{\perp}$. Of the four conditionals $C_1 - C_4$, all but the last one also satisfies this additional criterion (see [13]).

[1] G. Birkhoff and J. von Neumann, Ann. Mathematics **37**, 823 [1936].

[2] F. Kamber, Zweiwertige Wahrscheinlichkeitsfunktionen auf orthokomplementaren Verbänden, Math. Ann. **158**, 158 [1965].

[3] J. Jauch, Foundations of Quantum Mechanics, Addison-Wesley, New York 1968.

[4] J. Jauch and C. Piron, What is Quantum Logic?, in P. Freund et al. (eds.), Quanta, University of Chicago Press, Chicago 1970.

[5] H. Kunsemüller, Zur Axiomatik der Quantenlogik, Philosophia Naturalis **8**, 363 [1964].

[6] P. Mittelstaedt, Z. Naturforsch. **25a**, 1773 [1970].

[7] P. Mittelstaedt, Z. Naturforsch. **27a**, 1358 [1972].

[8] M. Nakamura, Kodai Math. Sem. Rep. **9**. 158 [1957].

[9] S. Holland, The Current Interest in Orthomodular Lattices, in J. C. Abbott (ed.), Trends in Lattice Theory, Van Nostrand Reinhold, New York 1970, pp. 41–126.

[10] D. Foulis, Portugaliae Mathematica **21**, 65 [1962].

[11] J. Pool, Comm. in Math. Physics **9**, 212 [1968].

[12] Th. Skolem, Untersuchungen über die Axiome des Klassenkalküls und über Produktions- und Summations-Probleme, welche gewisse Klassen von Aussagen betreffen, Videnskapsselskapets Skrifter, I, Mat.-Nat. Klasse, 1919, no. 3, 37.

[13] G. Hardegree, Material Conditionals in Orthomodular Lattices, I, Notre Dame Journal of Formal Logic, in press.

[14] J. Kotas, Studia Logica **31**, 17 [1967].

[15] G. Hardegree, Synthese **29**, 63 [1974].

[16] P. Finch, Bull. Australian Mathematical Society **2, 101** [1970].

[17] L. Herman, E. Marsden, and R. Piziak, Notre Dame Journal of Formal Logic **16**, 305 [1975].